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KRUSKAL–KATONA TYPE THEOREMS FOR CLIQUE COMPLEXES ARISING FROM CHORDAL AND STRONGLY CHORDAL GRAPHS

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Abstract. A forest is the clique complex of a strongly chordal graph and a quasi-forest is the clique complex of a chordal graph. Kruskal–Katona type theorems for forests, quasi-forests, pure forests and pure quasi-forests will be presented.

Introduction

Recently, in commutative algebra, the forest ([5]) and the quasi-forest ([17] and [9]) have been extensively studied. Each of these concepts is, however, well known in combinatorics ([14]). In fact, a forest is the clique complex of a strongly chordal graph and a quasi-forest is the clique complex of a chordal graph. (A chordal graph is a finite graph for which every cycle of length > 3 has a chord. A strongly chordal graph is a chordal graph for which every cycle of even length ≥ 6 has a chord that joins two vertices of the cycle with an odd distance > 1 in the cycle.)

Besides the celebrated g-conjecture for spheres ([16, pp. 75–76]), one of the most important open problems in the study of f-vectors of simplicial complexes is the classification of f-vectors of flag complexes. (A flag complex is the clique complex of a finite graph.) Works in the reserach topic include [3], [6], [7], [8] and [15]. On the other hand, the study of f-vectors of clique complexes of chordal graphs was done in [4], [12] and [13].

The purpose of the present paper is to give a Kruskal–Katona type theorem for forests and quasi-forests (Theorem 1.1) as well as a Kruskal–Katona type theorem for pure forests and pure quasi-forests (Theorem 1.2). These theorems will be proved in Section 2. We then show in Section 3 that the f-vector of a pure quasi-forest is unimodal.

1. Kruskal–Katona type theorems

Let $[n] = \{1, \ldots, n\}$ be the vertex set and Δ a simplicial complex on $[n]$. Thus Δ is a collection of subsets of $[n]$ with the properties that (i) $\{i\} \in \Delta$ for each $i \in [n]$ and (ii) if $F \in \Delta$ and $G \subset F$, then $G \in \Delta$. Each element $F \in \Delta$ is a face of Δ. Let $d = \max\{|F| : F \in \Delta\}$, where $|F|$ is the cardinality of F. Then $\dim \Delta$, the dimension of Δ, is $d - 1$. A facet is a maximal face of Δ under inclusion. We write $\mathcal{F}(\Delta)$ for the set of facets of Δ. A simplicial complex is called pure if all facets have the same cardinality. Let f_i denote the number of faces F with $|F| = i + 1$. The vector $f(\Delta) = (f_0, f_1, \ldots, f_{d-1})$ is called the f-vector of Δ. In particular $f_0 = n$. If

$\{F_{i_1}, \ldots, F_{i_q}\}$ is a subset of $\mathcal{F}(\Delta)$, then we write $\langle F_{i_1}, \ldots, F_{i_q}\rangle$ for the subcomplex of Δ whose faces are those faces F of Δ with $F \subset F_{i_j}$ for some $1 \leq j \leq q$.

A facet F of a simplicial complex Δ is called a *leaf* if there is a facet $G \neq F$ of Δ, called a *branch* of F, such that $H \bigcap F \subset G \bigcap F$ for all facets H of Δ with $H \neq F$. A *quasi-forest* is a simplicial complex Δ which enjoys an ordering $F_1, F_2, \ldots, F_s$ of the facets of Δ, called a *leaf order*, such that for each $1 < j \leq s$ the facet F_j is a leaf of the subcomplex $\langle F_1, \ldots, F_{j-1}, F_j\rangle$ of Δ. A *quasi-tree* is a quasi-forest which is connected. A *forest* is a simplicial complex Δ which enjoys the property that for every subset $\{F_{i_1}, \ldots, F_{i_q}\}$ of $\mathcal{F}(\Delta)$ the subcomplex $\langle F_{i_1}, \ldots, F_{i_q}\rangle$ of Δ has a leaf. A *tree* is a forest which is connected.

We now come to Kruskal–Katona type theorems for forests, quasi-forests, pure forests and pure quasi-forests.

Theorem 1.1. *Given a finite sequence $(f_0, f_1, \ldots, f_{d-1})$ of integers with each $f_i > 0$, the following conditions are equivalent:*

(i) there is a quasi-forest Δ of dimension $d-1$ with $f(\Delta) = (f_0, f_1, \ldots, f_{d-1})$;

(ii) there is a forest Δ of dimension $d-1$ with $f(\Delta) = (f_0, f_1, \ldots, f_{d-1})$;

(iii) the sequence $(c_1, \ldots, c_d)$ defined by the formula

$$\sum_{i=0}^{d} f_{i-1}(x-1)^i = \sum_{i=0}^{d} c_i x^i, \tag{1}$$

where $f_{-1} = 1$, satisfies $\sum_{i=k}^{d} c_i > 0$ for each $1 \leq k \leq d$.

(iv) the sequence $(b_1, \ldots, b_d)$ defined by the formula

$$\sum_{i=1}^{d} f_{i-1}(x-1)^{i-1} = \sum_{i=1}^{d} b_i x^{i-1} \tag{2}$$

is positive, i.e., $b_i > 0$ for $1 \leq i \leq d$.

Theorem 1.2. *Given a finite sequence $(f_0, f_1, \ldots, f_{d-1})$ of integers with each $f_i > 0$, the following conditions are equivalent:*

(i) there is a pure quasi-forest Δ of dimension $d-1$ with $f(\Delta) = (f_0, f_1, \ldots, f_{d-1})$;

(ii) there is a pure forest Δ of dimension $d-1$ with $f(\Delta) = (f_0, f_1, \ldots, f_{d-1})$;

(iii) the sequence $(c_1, \ldots, c_d)$ defined by (1) satisfies $\sum_{i=k}^{d} c_i > 0$ for each $1 \leq k \leq d$ and $c_i \leq 0$ for each $1 \leq i < d$.

(iv) the sequence $(b_1, \ldots, b_d)$ defined by the formula (2) satisfies $0 < b_1 \leq b_2 \leq \cdots \leq b_d$.

In Section 2, after preparing Lemmata 2.1, 2.2 and 2.3, we will prove both of Theorems 1.1 and 1.2 simultaneously.

2. f-vectors of forests and quasi-forests

We begin with

Lemma 2.1. *Let Δ be a quasi-forest on $[n]$ with $s+1$ facets and $F_{s+1}, F_s, \ldots, F_1$ its leaf order. For each $1 \leq j \leq s$ we write G_j for a branch of the leaf F_j in the subcomplex $\langle F_{s+1}, F_s, \ldots, F_j\rangle$ of Δ. Let $\delta_j = |F_j|$ and $e_j = |F_j \bigcap G_j|$. Let $\dim \Delta = d-1$ and let $f(\Delta) = (f_0, f_1, \ldots, f_{d-1})$ be the f-vector of Δ.*

(a) One has

$$\sum_{i=0}^{d} f_{i-1}x^i = \sum_{j=1}^{s+1}(1+x)^{\delta_j} - \sum_{j=1}^{s}(1+x)^{e_j}, \tag{3}$$

where $f_{-1} = 1$.

(b) Let $k_1 \cdots k_s k_{s+1}$ *be a permutation of* $[s+1]$ *with* $0 < \delta_{k_1} \leq \cdots \leq \delta_{k_s} \leq \delta_{k_{s+1}} = d$ *and* $\ell_1 \cdots \ell_s$ *a permutation of* $[s]$ *with* $0 \leq e_{\ell_1} \leq \cdots \leq e_{\ell_s}$. *Then* $e_{\ell_j} < \delta_{k_j}$ *for all* $1 \leq j \leq s$.

Proof. (a) Let $s = 0$. Then $\delta_1 = d$, $f_{i-1} = \binom{d}{i}$ and $\sum_{i=0}^{d} f_{i-1}x^i = (1+x)^{\delta_1}$. Let $s \geq 1$ and $\Delta' = \langle F_{s+1}, F_s, \ldots, F_2 \rangle$. Let $\dim \Delta' = d' - 1$ and $f(\Delta') = (f'_0, f'_1, \ldots, f'_{d'-1})$. Since Δ' is a quasi-forest, working by induction on the number of facets, it follows that $\sum_{i=0}^{d'} f'_{i-1}x^i = \sum_{j=2}^{s+1}(1+x)^{\delta_j} - \sum_{j=2}^{s}(1+x)^{e_j}$. Now, the number of faces F of Δ with $F \notin \Delta'$ and with $|F| = i$ is $\binom{\delta_1}{i} - \binom{e_1}{i}$. Hence

$$\sum_{i=0}^{d} f_{i-1}x^i = \sum_{i=0}^{d'} f'_{i-1}x^i + ((1+x)^{\delta_1} - (1+x)^{e_1}),$$

as desired.

(b) Let $k_p = 1$ and $\ell_q = 1$. Then $e_{\ell_q} < \delta_{k_p}$. Since $\langle F_{s+1}, F_s, \ldots, F_2 \rangle$ is a quasi-forest, working by induction on the number of facets, it follows that (i) in case of $p \leq q$, one has $e_{\ell_j} < \delta_{k_j}$ for each $1 \leq j < p$ and for each $q < j \leq s$, and $e_{\ell_j} \leq e_{\ell_q} < \delta_{k_p} \leq \delta_{k_j}$ for each $p \leq j \leq q$, and that (ii) in case of $q < p$, one has $e_{\ell_j} < \delta_{k_j}$ for each $1 \leq j < q$ and for each $p < j \leq s$, and $e_{\ell_j} \leq e_{\ell_{j+1}} < \delta_{k_j} \leq \delta_{k_{j+1}}$ for each $q \leq j < p$. Hence $e_{\ell_j} < \delta_{k_j}$ for all $1 \leq j \leq s$. □

Lemma 2.2. *Given a quasi-forest of dimension* $d-1$ *with* $s+1$ *facets and with* f*-vector* $f(\Delta) = (f_0, f_1, \ldots, f_{d-1})$, *there exist finite sequences* $(\delta_1, \ldots, \delta_t, \delta_{t+1})$ *and* $(e_1, \ldots, e_t)$ *of integers, where* $0 < t \leq s$ *and where* $\delta_i \neq e_j$ *for all* i *and* j, *satisfying*

$$0 < \delta_1 \leq \cdots \leq \delta_t \leq \delta_{t+1} = d, \qquad 0 \leq e_1 \leq \cdots \leq e_t < d$$

and

$$e_j < \delta_j, \quad 1 \leq j \leq t$$

which enjoys the formula

$$\sum_{i=0}^{d} f_{i-1}x^i = \sum_{j=1}^{t+1}(1+x)^{\delta_j} - \sum_{j=1}^{t}(1+x)^{e_j},$$

where $f_{-1} = 1$.

Proof. In Lemma 2.1 (b), in case that $\delta_{k_a} = e_{\ell_b}$ for some a and b, one has $a < b$ and $e_{\ell_j} < \delta_{k_{j+1}}$ for all $a \leq j < b$. Hence we can replace $\delta_{k_1}, \ldots, \delta_{k_{s+1}}$ and $e_{\ell_1}, \ldots, e_{\ell_s}$ with $\delta_{k_1}, \ldots, \delta_{k_{a-1}}, \delta_{k_{a+1}}, \ldots, \delta_{k_s}, \delta_{k_{s+1}}$ and $e_{\ell_1}, \ldots, e_{\ell_{b-1}}, e_{\ell_{b+1}}, \ldots, e_{\ell_s}$. □

Lemma 2.3. *Let* $(\delta_1, \ldots, \delta_s, \delta_{s+1})$ *and* $(e_1, \ldots, e_s)$ *be sequences of integers, where* $s \geq 0$ *and where* $\delta_i \neq e_j$ *for all* i *and* j, *satisfying*

$$0 < \delta_1 \leq \cdots \leq \delta_s \leq \delta_{s+1} = d, \qquad 0 \leq e_1 \leq \cdots \leq e_s < d$$

and

$$e_j < \delta_j, \quad 1 \leq j \leq s.$$

Then there is a forest Δ *on* $[n]$*, where* $n = \sum_{j=1}^{s+1} \delta_j - \sum_{j=1}^{s} e_j$*, of dimension* $d-1$ *such that the* f*-vector* $f(\Delta) = (f_0, f_1, \ldots, f_{d-1})$ *of* Δ *satisfies (3).*

Proof. First, we construct the subsets $F_1, \ldots, F_s, F_{s+1}$ of $[n]$, where $|F_j| = \delta_j$ for each $1 \leq j \leq s+1$ and where $|F_j \bigcap F_{j+1}| = e_j$ for each $1 \leq j \leq s$. Let $F_{s+1} = \{n-d+1, n-d+2, \ldots, n\}$. If we obtain the facet $F_j = \{q_1, q_2, \ldots, q_{\delta_j - 1}, q_{\delta_j}\}$, where $1 \leq q_1 < q_2 < \cdots < q_{\delta_j} \leq n$, then the facet F_{j-1} is defined to be

$$F_{j-1} = \{q_1 - (\delta_{j-1} - e_{j-1}), \ldots, q_1 - 2, q_1 - 1, q_{\delta_j - e_{j-1} + 1}, \ldots, q_{\delta_j - 1}, q_{\delta_j}\}.$$

A crucial property of $F_1, \ldots, F_s, F_{s+1}$ is that

$$F_j \bigcap F_{s+1} = \cdots = F_j \bigcap F_{j+2} = F_j \bigcap F_{j+1} \tag{4}$$

for each $1 \leq j \leq s$.

Now, write Δ for the simplicial complex on $[n]$ of dimension $d-1$ with $\mathcal{F}(\Delta) = \{F_1, \ldots, F_s, F_{s+1}\}$. It follows from the property (4) that Δ is a quasi-forest with $F_{s+1}, F_s, \ldots, F_1$ its leaf order. In addition, for each $1 \leq j \leq s$, in the quasi-forest $\langle F_{s+1}, F_s, \ldots, F_j \rangle$ the facet F_{j+1} is a branch of the leaf F_j. Since $|F_j| = \delta_j$ for each $1 \leq j \leq s+1$ and $|F_j \bigcap F_{j+1}| = e_j$ for each $1 \leq j \leq s$, Lemma 2.1 guarantees that the f-vector $f(\Delta) = (f_0, f_1, \ldots, f_{d-1})$ of Δ satisfies (3).

Finally, we claim that Δ is a forest. Let $1 \leq j_1 < j_2 < \cdots < j_q \leq s+1$ and $\Gamma = \langle F_{j_1}, \ldots, F_{j_q} \rangle$. Then

$$F_{j_1} \bigcap F_{j_q} = \cdots = F_{j_1} \bigcap F_{j_3} = F_{j_1} \bigcap F_{j_2}.$$

Hence F_{j_1} is a leaf of Γ with F_{j_2} its branch. $\square$

We now prove both of Theorems 1.1 and 1.2 simultaneously.

Proof. ((ii) $\Rightarrow$ (i)) Since a (resp. pure) forest is a (resp. pure) quasi-forest, the f-vectors of (resp. pure) forests coincide with the f-vectors of (resp. pure) quasi-forests.

((i) $\Rightarrow$ (iii)) Let Δ be a quasi-forest of dimension $d-1$ with $s+1$ facets and $f(\Delta) = (f_0, f_1, \ldots, f_{d-1})$ its f-vector. With the same notation as in Lemma 2.2, it follows that

$$\sum_{i=0}^{d} c_i x^i = \sum_{j=1}^{t+1} x^{\delta_j} - \sum_{j=1}^{t} x^{e_j}.$$

Thus

$$\sum_{i=k}^{d} c_i = |\{j : \delta_j \geq k\}| - |\{j : e_j \geq k\}| > 0$$

for each $1 \leq k \leq d$, as desired.

If, in addition, Δ is pure, then Lemma 2.1 guarantees the existence of a sequence $(e_1, \ldots, e_s)$ of integers with $0 \leq e_1 \leq \cdots \leq e_s < d$ such that

$$\sum_{i=0}^{d} c_i x^i = (s+1)x^d - \sum_{j=1}^{s} x^{e_j}.$$

Thus $\sum_{i=k}^{d} c_i > 0$ for each $1 \leq k \leq d$ and $c_i \leq 0$ for each $1 \leq k < d$.

((iii) $\Rightarrow$ (ii)) Let $s+1 = \sum_{c_i>0} c_i$. Since $\sum_{i=0}^{d} c_i = f_{-1} = 1$, one has $\sum_{c_i<0}(-c_i) = s$. Since $\sum_{i=1}^{d} c_i > 0$ and $\sum_{i=0}^{d} c_i = f_{-1} = 1$, one has $c_0 \leq 0$. Write

$$\sum_{i=0}^{d} c_i x^i = \sum_{j=1}^{s+1} x^{\delta_j} - \sum_{j=1}^{s} x^{e_j},$$

where $e_i \neq \delta_j$ for all i and j, and where

$$0 < \delta_1 \leq \cdots \leq \delta_s \leq \delta_{s+1} = d, \qquad 0 \leq e_1 \leq \cdots \leq e_s < d.$$

Suppose $\sum_{i=k}^{d} c_i > 0$ for each $1 \leq k \leq d$. We claim $e_j < \delta_j$ for each $1 \leq j \leq s$. To see why this is true, let j denote the biggest integer $\leq s$ for which $\delta_j < e_j$. Then $\sum_{i=e_j}^{d} c_i = 0$, a contradiction.

Now, it turns out that the sequences $(\delta_1, \ldots, \delta_s, \delta_{s+1})$ and $(e_1, \ldots, e_s)$ enjoy the properties required in Lemma 2.3. Thus there exists a forest Δ of dimension $d-1$ whose f-vector $f(\Delta)$ satisfies (3). In other words, the f-vector $f(\Delta)$ must coincide with the given sequence $(f_0, f_1, \ldots, f_{d-1})$.

If, in addition, $c_i \leq 0$ for each $1 \leq i < d$, then $\delta_j = d$ for all $1 \leq j \leq s+1$. Hence the forest which is constructed in the proof of Lemma 2.3 is pure.

((iii) $\Leftrightarrow$ (iv)) Let $b_k = \sum_{i=k}^{d} c_i$ for $k = 0, 1, \ldots, d$. In other words, the sequence $(b_0, b_1, \ldots, b_d)$ is defined by the formula

$$\sum_{i=0}^{d} c_i x^i = \sum_{i=1}^{d} b_i x^{i-1}(x-1) + b_0.$$

Since $b_0 = c_0 + \cdots + c_d = f_{-1}$, it follows from (1) that $(b_1, \ldots, b_d)$ satisfies (2). It is now clear that (iii) is equivalent to (iv), as desired. $\square$

Remark 2.4. Let Δ be a simplicial complex on $[n]$ of dimension $d-1$ and $f(\Delta) = (f_0, f_1, \ldots, f_{d-1})$ its f-vector. Recall from [1], [11] and [16] that the h-vector $h(\Delta) = (h_0, h_1, \ldots, h_d)$ of Δ is defined by the formula

$$\sum_{i=0}^{d} f_{i-1}(x-1)^{d-i} = \sum_{i=0}^{d} h_i x^{d-i},$$

or equivalently, by the formula

$$\sum_{i=0}^{d} h_i x^i (1+x)^{d-i} = \sum_{i=0}^{d} f_{i-1} x^i. \tag{5}$$

In particular $h_0 = 1$ and $h_1 = n - d$. It follows from (1) and (5) that

$$\sum_{i=0}^{d} c_i x^i = \sum_{i=0}^{d} h_i x^{d-i}(x-1)^i.$$

Hence

$$c_i = \sum_{j=0}^{d} (-1)^{d-i} \binom{j}{d-i} h_j.$$

Consequently, for each $k = 1, 2, \ldots, d$, one has

$$b_k = \sum_{i=k}^{d} c_i = \sum_{j=0}^{d} \left\{ \sum_{i=k}^{d} (-1)^{d-i} \binom{j}{d-i} h_j \right\} = 1 + \sum_{j=1}^{d} (-1)^{d-k} \binom{j-1}{d-k} h_j.$$

3. Unimodality of f-vectors

A finite sequence $(a_1, a_2, \ldots, a_N)$ of integers with each $a_i > 0$ is called *unimodal* if $a_0 \leq \cdots \leq a_j \geq a_{j+1} \geq \cdots \geq a_N$ for some $0 \leq j \leq N$. We recall the following well-known

Lemma 3.1. *Let d and e be integers with $0 \leq e < d$ and define the sequence $(a_0, a_1, \ldots, a_\delta)$ by the formula $\sum_{i=0}^{\delta} a_i x^i = (1+x)^d - (1+x)^e$. Then*

$$a_0 \leq a_1 \leq \cdots \leq a_{[(d+1)/2]} \geq a_{[(d+1)/2]+1} \geq \cdots \geq a_d.$$

Theorem 3.2. *The f-vector $f(\Delta) = (f_0, f_1, \ldots, f_{d-1})$ of a pure quasi-forest Δ of dimension $d-1$ is unimodal.*

Proof. Let Δ be a pure quasi-forest of domension $d-1$ with $s+1$ facets and $f(\Delta) = (f_0, f_1, \ldots, f_{d-1})$ its f-vector. Lemma 2.1 says that there is a sequence $(e_1, \ldots, e_s)$ of integers with $0 \leq e_1 \leq \cdots \leq e_s < d$ such that

$$\sum_{i=0}^{d} f_{i-1} x^i = (1+x)^d + \sum_{j=1}^{s} ((1+x)^d - (1+x)^{e_j}).$$

By using Lemma 3.1 it follows that

$$f_{-1} \leq f_0 \leq \cdots \leq f_{[(d+1)/2]-1} \geq f_{[(d+1)/2]} \geq \cdots \geq f_{d-1}.$$

Hence $(f_0, f_1, \ldots, f_{d-1})$ is unimodal. $\square$

The f-vector of a pure simplicial complex is not necessarily unimodal. In fact, there is a simplicial convex polytope such that the f-vector of its boundary complex is not unimodal ([2]). On the other hand, it is proved in [10] that the f-vector $f(\Delta) = (f_0, f_1, \ldots, f_{d-1})$ of a pure simplicial complex Δ of dimension $d-1$ satisfies

$$f_i \leq f_{d-2-i}, \quad -1 \leq i \leq [d/2] - 1,$$

together with

$$f_{-1} \leq f_0 \leq \cdots \leq f_{[d/2]-1}.$$

References

[1] W. Bruns and J. Herzog, "Cohen–Macaulay rings," Revised Edition, Cambridge University Press, 1996.

[2] A Björner, The unimodality conjecture for convex polytopes, *Bull. Amer. Math. Soc.* **4** (1981), 187 – 188.

[3] R. Charney and M. Davis, The Euler characteristic of a nonpositively curved, piecewise Euclidean manifold, *Pacific J. Math.* **171** (1995), 117 – 137.

[4] J. Eckhoff, Über kombinatorisch-geometrische Eigenschaften von Komplexen und Familien konvexer Mengen, *J. Reine Angew. Math.* **313** (1980), 171 – 188.

[5] S. Faridi, Cohen–Macaulay properties of square-free monomial ideals, *J. Combin. Theory, Ser. A* **109** (2005), 299 – 329.

[6] D. Ferrarello and R. Fröberg, The Hilbert series of the clique complex, *Graphs Combin.* **21** (2005), 401 – 405.
[7] A. Frohmader, Face vectors of flag complexes, arXiv:math/0605673, preprint.
[8] S. R. Gal, Real root conjecture fails for five- and higher-dimensional spheres, *Discrete Comput. Geom.* **34** (2005), 269 – 284.
[9] J. Herzog, T. Hibi and X. Zheng, Dirac's theorem on chordal graphs and Alexander duality, *European J. Combin.* **25** (2004), 949 – 960.
[10] T. Hibi, What can be said about pure O-sequences?, *J. Combin. Theory, Series A* **50** (1989), 319 – 322.
[11] T. Hibi, "Algebraic Combinatorics on Convex Polytopes," Carslaw Publications, Glebe, N.S.W., Australia, 1992.
[12] G. Kalai, A characterization of f-vectors of families of convex sets in $\mathbb{R}$, Part I: Necessity of Eckhoff's conditions, *Israel J. Math.* **48** (1984), 175 – 195.
[13] G. Kalai, A characterization of f-vectors of families of convex sets in $\mathbb{R}$, Part II: Sufficiency of Eckhoff's conditions, *J. Comb. Theory Ser. A* **41** (1986), 167 – 188.
[14] T. A. McKee, F. R. McMorris, "Topics in Intersection Graph Theory," Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1999.
[15] P. Renteln, The Hilbert series of the face ring of a flag complex, *Graphs Combin.* **18** (2002), 605 – 619.
[16] R. Stanley, "Combinatorics and Commutative Algebra," Second Edition, Birkhäuser, 1995.
[17] X. Zheng, Resolutions of Facet Ideals, *Comm. Algebra* **32** (2004), 2301 – 2324.

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